On the Multiplication and Involution of Semi-convergent Series.

By Florian Cajori.

§1. In the raising of an absolutely convergent series by Cauchy's multiplication rule to any positive integral power, no tests of the convergence of the product-series are needed, but the case of semi-convergent series is more difficult. Theorems on the convergence of the product of two semi-convergent series have been given by Abel (Crelle's Journal, vol. I, pp. 311–339); A. Pringsheim (Math. Ann., vol. XXI, p. 327; XXVI, p. 157), and A. Voss (Math. Ann., vol. XXIV, p. 42). In the Am. Jour. of Math., XV, p. 339, the writer generalized Voss's results to the following effect: *The necessary and sufficient conditions that the product of two semi-convergent series Σa_n and Σb_n, of which one, say Σa_n, becomes absolutely convergent on associating its terms into groups having the same number p of terms, converge towards the product of the sums of the two given series, are that the n^{th} term and all succeeding terms of the product-series shall approach the limit zero as n increases indefinitely, and that the following relation be satisfied:**

$$\lim_{m=\infty} \sum_{i=0}^{i=m-1} \{ b_{pi+2}\, a_{pm-pi-1} + b_{pi+3}\,(a_{pm-pi-2} + a_{pm-pi-1}) + \cdots \\ + b_{pi+p}\,(a_{pm-pi-p+1} + \cdots + a_{pm-pi-1}) \} = 0. \quad \mathrm{I}$$

In this equation $pm = n$. In case of powers of semi-convergent series higher than the second, the tests offered by the theorems referred to are, as a rule, practically inoperative on account of the complexity of the expressions involved.

The search for expeditious tests on the applicability of Cauchy's multiplica-

* In the Bull. of the Am. Math. Soc., 2d series, vol. I, pp. 180–183, the writer has deduced the necessary and sufficient conditions for convergence in the more general case when the number of terms in the various groups is not necessarily the same.

tion rule to powers of semi-convergent series higher than the second power, has given rise to the following investigation which begins with alternating semi-convergent series and ends with certain trigonometric series.

§2. If the signs of the terms of each of two series are alternately plus and minus, then the product-series has its terms alternately plus and minus, and all the individual products, $a_{n-x} b_x$, which enter into the composition of any one term, $(a_0 b_n + a_1 b_{n-1} + \ldots . + a_n b_0)$, in the product-series, have like signs in that term. This theorem is easily verified.

§3. If the signs of the terms in each of two semi-convergent series are alternately plus and minus, and if one of the two series becomes absolutely convergent on associating its terms into groups of two terms each, then conditions I are satisfied and the product-series is convergent, whenever the n^{th} term and all succeeding terms of the product-series approach the limit zero as n increases indefinitely.

Proof.—When $p = 2$, then formula I demands that the sum of certain products, $a_{n-x+1} b_x$, which enter into the composition of the $(n+2)^{\text{th}}$ term, $(a_{n+1} b_0 + a_n b_1 + \ldots . + a_0 b_{n+1})$, shall approach the limit zero as n increases indefinitely. Now, *if* the entire $(n+2)^{\text{th}}$ term approaches the limit zero, then, since all the constituents, $a_{n-x+1} b_x$, have like signs (§2), it follows at once that the sum of any selected number of those constituents must approach zero as a limit, and formula I must be satisfied. Hence the examination of formula I becomes superfluous in this case.

§4. *The square of the semi-convergent series*

$$1 - \frac{1}{2^r} + \frac{1}{3^r} - \frac{1}{4^r} + \ldots . \pm \frac{1}{n^r}, \qquad (0 < r \overline{\gtrless} 1). \qquad \text{II}$$

From §2 it follows that each term in any of the powers of this series has the same absolute value as the corresponding term in the same power of the series

$$1 + \frac{1}{2^r} + \frac{1}{3^r} + \frac{1}{4^r} + \ldots . + \frac{1}{n^r}. \qquad \text{III}$$

The latter series may be written thus:

$$1 + \left(\frac{1}{2^r} + \frac{1}{3^r}\right) + \left(\frac{1}{4^r} + \frac{1}{5^r} + \frac{1}{6^r} + \frac{1}{7^r}\right) + \ldots .$$
$$+ \left(\frac{1}{2^{mr}} + \frac{1}{(2^m + 1)^r} + \ldots . + \frac{1}{(2^{m+1} - 1)^r}\right). \qquad \text{IV}$$

Here
$$n = 2^{m+1} - 1. \tag{V}$$
Evidently
$$\left(\frac{1}{2^{rm}} + \frac{1}{(2^m+1)^r} + \dots + \frac{1}{(2^{m+1}-1)^r}\right) \begin{matrix} < \dfrac{2^m}{2^{mr}} \\ > \dfrac{2^m}{2^{(m+1)r}} \end{matrix} .$$
Hence the sum in IV is
$$< 1 \quad + 2^{1-r} + 2^{(1-r)2} + \dots + 2^{(1-r)m} \quad , \text{ or } < \frac{2^{(1-r)(m+1)} - 1}{2^{1-r} - 1}, \tag{VI}$$
and the sum in IV is
$$> 2^{-r} + 2^{1-2r} + 2^{2-3r} + \dots + 2^{m-(m+1)r}, \text{ or } > \frac{2^{-r}\left[2^{(1-r)(m+1)} - 1\right]}{2^{1-r} - 1}. \tag{VII}$$

For brevity, let the series III be represented by $a_0 + a_1 + a_2 + \dots + a_n$. The $2n^{\text{th}}$ term of the square of this series is $(a_{2n-1}a_0 + a_{2n-2}a_1 + \dots + a_0a_{2n-1}) = 2(a_{2n-1}a_0 + a_{2n-2}a_1 + \dots + a_na_{n-1}) < 2(a_0 + a_1 + \dots + a_{n-1})\,a_n$. From this and from VI it follows that in the square of series III, the $2n^{\text{th}}$ term, which we designate by b_{2n-1}, is less than
$$\frac{2^{(1-r)(m+1)+1}}{\left[2^{1-r} - 1\right]\, 2^{(m+1)r}}. \tag{VIII}$$

Fraction VIII approaches the limit zero as n increases indefinitely whenever $r > \frac{1}{2}$. Hence the $2n^{\text{th}}$ term in the square of III approaches zero whenever $r > \frac{1}{2}$. The same conclusion is reached for the $(2n-1)^{\text{th}}$ term. Since series II becomes absolutely convergent by taking $p = 2$, it follows from the above and from §3 that the square of II is convergent and equal to U^2, U being the sum of series II, whenever $r > \frac{1}{2}$.

When $r \overline{\gtrless} \frac{1}{2}$ it is well known that the square of series II is divergent.*

§5. *The cube of series* II. The $(4n+1)^{\text{th}}$ term of the product of $a_0 + a_1 + \dots + a_n$ and of $b_0 + b_1 + b_2 + \dots + b_n$ has the form $(a_{4n}b_0 + a_{4n-1}b_1 + \dots + a_0b_{4n})$. This is less than $(b_0 + b_1 + \dots + b_{2n})\,a_{2n} + (a_0 + a_1 + \dots + a_{2n-1})\,b_y$, on the supposition that $a_0 + a_1 + \dots$ stand for series III, that $b_0 + b_1 + b_2 + \dots$ stand for the square of III, and that b_y be the greatest term in the succession of terms $b_{2n+1}, b_{2n+2}, \dots b_{4n}$.

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* See Cauchy, Cours d'Analyse, vol. I, p. 149.

Now

$$b_{2n-1} < \frac{2^{(1-r)(m+1)+1}}{[2^{1-r}-1]\,2^{(m+1)r}}$$

and

$$b_{4n} < \frac{2^{(1-r)(m+2)+1}}{[2^{1-r}-1][2^{m+2}-1]^r};$$

hence

$$b_y < \frac{2^{(1-r)(m+2)+1}}{[2^{1-r}-1]\,2^{(m+1)r}}$$

and

$$(a_0+a_1+\ldots\ldots+a_{2n-1})\,b_y < \frac{2^{2(1-r)(m+2)+1}}{[2^{1-r}-1]^2\,2^{(m+1)r}}. \qquad \text{IX}$$

Find values for b_{2n-2}, b_{2n}, analogous to the values for b_{2n-1} and b_{4n}, and we then get the relations

$$\frac{b_{2n-2}}{a_{2n-2}} < \frac{2^{(1-r)(m+1)+1}\,[2^{m+2}-3]^r}{[2^{1-r}-1][2^{m+1}-1]^r},$$

$$\frac{b_{2n-1}}{a_{2n-1}} < \frac{2^{(1-r)(m+1)+1}\,[2^{m+2}-2]^r}{[2^{1-r}-1]\,2^{(m+1)r}},$$

$$\frac{b_{2n}}{a_{2n}} < 2\left\{\frac{2^{(1-r)(m+1)}}{2^{1-r}-1}+\frac{1}{2^{(m+1)r}}\right\}\frac{[2^{m+2}-1]^r}{2^{(m+1)r}}.$$

Of the right-hand members of these three inequalities, the third exceeds each of the two others, and therefore exceeds the right-hand member of a similar inequality formed in connection with any preceding terms of the two series $a_0+a_1+a_2+\ldots\ldots$ and $b_0+b_1+b_2+\ldots\ldots$ In place of this third fraction take another still larger, but somewhat simpler in form, and we have then

$$(b_0+b_1+\ldots\ldots+b_{2n}) < (a_0+a_1+\ldots\ldots+a_{2n})\,\frac{2^{(1-r)(m+2)+1}\,[2^{m+2}-1]^r}{[2^{1-r}-1]\,2^{(m+1)r}}$$

and

$$(b_0+b_1+\ldots\ldots+b_{2n})\,a_{2n} < \frac{2^{2(1-r)(m+2)+1}}{[2^{1-r}-1]^2\,2^{(m+1)r}}. \qquad \text{X}$$

Combining IX and X, we conclude that the $(4n+1)^{\text{th}}$ term of the cube of series II is smaller in absolute value than

$$\frac{2^{2(1-r)(m+2)+2}}{[2^{1-r}-1]^2\,2^{(m+1)r}}. \qquad \text{XI}$$

As m increases indefinitely, fraction XI approaches the limit zero, when $r>\frac{2}{3}$. This same conclusion is reached for a similar fraction known to be greater than the $(4n-x+1)^{\text{th}}$ term [x being any value 1, 2, 3]. Hence the cube of II converges whenever $r>\frac{2}{3}$, (§3).

To determine the nature of the cube-series when $r < \frac{2}{3}$, observe that the $(2n+1)^{\text{th}}$ term of the square of series III, namely,

$$b_{2n} > (a_0 + a_1 + \ldots + a_n)\, a_{2n}, \text{ i. e.} > \frac{2^{-r}\left[2^{(1-r)(m+1)} - 1\right]}{\left[2^{1-r} - 1\right] 2^{(m+2)r}}$$

and

$$b_{4n} > (a_0 + a_1 + \ldots + a_n)\, a_{4n}, \text{ i. e.} > \frac{2^{-r}\left[2^{(1-r)(m+1)} - 1\right]}{\left[2^{1-r} - 1\right] 2^{(m+3)r}}.$$

Hence b_z, the smallest term in the succession $b_{2n}, b_{2n+1}, \ldots, b_{4n}$, is larger than the last fraction. The $(4n+1)^{\text{th}}$ term of the cube of series III is larger than $(a_0 + a_1 + \ldots + a_n)\, b_z$; that is, larger than the fraction

$$\frac{2^{-2r}\left[2^{(1-r)(m+1)} - 1\right]^2}{\left[2^{1-r} - 1\right]^2 2^{(m+3)r}}. \qquad \text{XII}$$

Whenever $r < \frac{2}{3}$, then this fraction does not approach the limit zero, as m increases indefinitely; nor does the $(4n+1)^{\text{th}}$ term of the cube of series II approach zero, for the reason that the constituent terms of which the $(4n+1)^{\text{th}}$ term is composed, are all alike in sign (§2), and the part of it represented by $(a_0 b_{4n} + a_1 b_{4n-1} + \ldots + a_{2n} b_{2n})$ is larger in absolute value than fraction XII. Hence the cube-series is divergent whenever $r < \frac{2}{3}$.

§6. *The q-power of series* II. The results obtained in the investigation of the convergence of the square-series and cube-series (see VIII and XI) are in conformity with the following general statement:

If $c_{2^{(q-1)}n}$ designate the $(2^{q-1}n + 1)^{\text{th}}$ term in the q-power of series III, then is

$$c_{2^{q-1}n} < \frac{2^{(1-r)(m+q-1)(q-1)+q-1}}{\left[2^{1-r} - 1\right]^{q-1} 2^{(m+1)r}}, \qquad \text{XIII}$$

and the $(2^{q-1}n - x + 1)^{\text{th}}$ term (where x may have any value $1, 2, \ldots, 2^{q-1}$) is less than a similar fraction whose limit vanishes under the same condition as does the limit of XIII. We proceed to show that this relation is true for any power whatever, by proving first that if it holds true for some one power q, it holds true for the power $q+1$. If it is true for the power q, then we may proceed as follows:

$$(a_0 + a_1 + \ldots + a_{2^{q-1}n}) < \frac{2^{(1-r)(m+q)}}{2^{1-r} - 1}.$$

If c_y represent the largest term in the succession $c_{2^{q-1}n}, c_{2^{q-1}n+1}, \ldots, c_{2^q n}$,

then, if we compare the inequality

$$c_{2^q n} < \frac{2^{(1-r)(m+q)(q-1)+q-1}}{\left[2^{1-r}-1\right]^{q-1} 2^{(m+1)r}} \qquad \text{XIII(a)}$$

(obtained from XIII by allowing the numerator to increase at a rate not slower than that demanded by relation V, while the denominator was kept constant) with the corresponding one for $c_{2^{q-1}n}$, we find that c_y must be less than the fraction in XIII(a). This is evident when we consider that for any assumed value r within the limits 0 and 1, the numerator never decreases as we pass to higher terms of the series, since m never decreases when n increases. Hence

$$(a_1 + a_2 + \ldots + a_{2^{q-1}n})\, c_y < \frac{2^{(1-r)(m+q)q+q-1}}{\left[2^{1-r}-1\right]^{q} 2^{(m+1)r}}. \qquad \text{XIV}$$

We have also

$$c_{2^{q-1}n} < a_{2^{q-1}n} \frac{2^{(1-r)(m+q)(q-1)+q-1}\left[2^{m+q}-2^{q-1}+1\right]^r}{\left[2^{1-r}-1\right]^{q-1} 2^{(m+1)r}}.$$

Since $c_{2^q n}$, $c_{2^{q-1}n}$, and all intervening terms (and therefore all the lower terms) are less than the fraction in XIII(a), it follows that the fraction in our last inequality is larger than the fraction in a similar inequality established in connection with any preceding terms of the two series $a_0 + a_1 + \ldots$ and $c_0 + c_1 + \ldots$. Hence

$$(c_0 + c_1 + \ldots + c_{2^{q-1}n})\, a_{2^{q-1}n} < \frac{2^{(1-r)(m+q)q+q-1}}{\left[2^{1-r}-1\right]^{q} 2^{(m+1)r}}. \qquad \text{XV}$$

By adding XIV and XV, we find the $(2^q n+1)^{\text{th}}$ term of the $(q+1)^{\text{th}}$ power of series III to be less than

$$\frac{2^{(1-r)(m+q)q+q}}{\left[2^{1-r}-1\right]^{q} 2^{(m+1)r}}.$$

By comparing this with a similar fraction which is greater than the $(2^{q-1}n+1)^{\text{th}}$ term of the $(q+1)^{\text{th}}$ power, we easily obtain a fraction which is greater than the $(2^q n - x + 1)^{\text{th}}$ term [where x may have any value $1, 2, \ldots, 2^q$], but which approaches the limit zero under the same conditions as does the fraction given above.

Thus it appears that if the relation XIII holds true for some one quantity q, it holds for $q+1$, but it is true for $q=3$ and therefore for any power. The fraction XIII approaches the limit zero, as m increases indefinitely, whenever $r > \frac{q-1}{q}$. Hence the series II can be raised to a power q, subject to the condi-

tion that $\frac{q-1}{q} < r$ (§3). We have thus established the theorem that *the semi-convergent series* $\sum \frac{(-1)^{n+1}}{n^r}$, $(0 < r \overline{\gtrless} 1)$, *will remain convergent if raised by Cauchy's multiplication rule to a positive integral power* q, *or to any lower positive integral power, whenever* $\frac{q-1}{q} < r$.

Proceeding to the investigation of the case $r < \frac{q-1}{q}$, we assume that the relation analogous to the inequality XII holds in the case of the q^{th} power, and then show that it holds true for the $(q+1)^{\text{th}}$ power. If it is true for the q^{th} power, then the $(2^{q-1}n+1)^{\text{th}}$ term of the q^{th} power of series III is larger than the quantity analogous to fraction XII, i. e.

$$c_{2^{q-1}n} > \frac{2^{-(q-1)r}\left[2^{(1-r)(m+1)}\right]^{q-1}}{\left[2^{1-r}-1\right]^{q-1}2^{(m+q)r}}. \tag{XVI}$$

We have also

$$c_{2^q n} > \frac{2^{-(q-1)r}\left[2^{(1-r)(m+1)}\right]^{q-1}}{\left[2^{1-r}-1\right]^{q-1}2^{(m+q+1)r}},$$

hence the smallest term c_z in $c_{2^{q-1}n}, c_{2^{q-1}n+1}, \ldots, c_{2^q n}$ is larger than the last fraction. Now the $(2^q n+1)^{\text{th}}$ term of the $(q+1)^{\text{th}}$ power of series III is larger than $(a_0 + a_1 + \ldots + a_n)\, c_z$, that is, larger than

$$\frac{2^{-qr}\left[2^{(1-r)(m+1)}-1\right]^q}{\left[2^{1-r}-1\right]^q\, 2^{(m+q+1)r}}. \tag{XVII}$$

Hence the inequality XVI holds true for any positive integral power of series III. But the fraction XVI does not approach the limit zero whenever $\frac{q-1}{q} > r$. Hence the $(2^{q-1}n+1)^{\text{th}}$ term of the q^{th} power of series II does not approach the limit zero as n increases indefinitely whenever $\frac{q-1}{q} > r$, and the power-series cannot be convergent. We have thus established the theorem that *the* q^{th} *power of the semi-convergent series* $\sum \frac{(-1)^{n+1}}{n^r}$, *obtained by Cauchy's multiplication rule, is divergent whenever* $\frac{q-1}{q} > r$.

§7. If we consider the two series $\sum \frac{(-1)^{n+1}}{n^r}$ and $\sum \frac{(-1)^{n+1}}{n^s}$ (where both r and s are > 0 and $\overline{\gtrless} 1$), we find by the method of §4 that the $(2n-1)^{\text{th}}$ term

of their product is numerically less than

$$\frac{2^{(1-s)(m+1)}}{\left[2^{1-s}-1\right]2^{(m+1)r}}+\frac{2^{(1-r)(m+1)}}{\left[2^{1-r}-1\right]2^{(m+1)s}}\,. \qquad \text{XVIII}$$

Both these fractions vanish as n increases indefinitely, whenever $r+s>1$. This same conclusion is reached for the $2n^{\text{th}}$ term of the product.

With the view of studying the case $r+s=1$, consider the $(2n-1)^{\text{th}}$ term, viz.

$$\frac{1}{(2n-1)^s}+\frac{1}{2^r(2n-2)^s}+\ldots\ldots+\frac{1}{n^r n^s}+\ldots\ldots+\frac{1}{(2n-1)^r}\,.$$

The terms of this series are all alike in sign. The first n terms continually decrease in absolute value from left to right, or else the last n terms continually increase in that direction. Hence the sum, either of the first n terms or of the last n terms, is numerically greater than $\frac{n}{n^{r+s}}$. Since this fraction is equal to unity for any value of n, it follows that the $(2n-1)^{\text{th}}$ term does not approach zero as a limit, and that the product-series diverges. *Thus* $r+s>1$ *is a necessary and sufficient condition for the applicability of Cauchy's rule to the multiplication, by one another, of the series* $\sum\frac{(-1)^{n+1}}{n^r}$ *and* $\sum\frac{(-1)^{n+1}}{n^s}$.

§8. The disjunctive criterion $r+s>1$ holds true in the more general case of two convergent series in which all the terms after a finite number of terms from the origin have the same absolute values as the corresponding terms in $\sum\frac{(-1)^{n+1}}{n^r}$ and $\sum\frac{(-1)^{n+1}}{n^s}$ respectively, and can be associated into groups with respect to their signs, so that in either series there is the same number of terms in all the groups, and this number is the same in both series: the order of progression of the signs in all the groups of one factor-series being either the opposite of what it is in all groups of the other factor-series, or the opposite with all signs reversed. For instance, if the terms in the groups of one factor-series have signs $+ + + - - + - -$, then the terms in the groups of the other factor-series must have either the signs $- - + - - + + +$ or the signs $+ + - + + - - -$. Observe that when the series are convergent there must be in each group as many terms with the sign $+$ as there are terms with

the — sign,* and that the two series, with the terms thus grouped, are absolutely convergent. The product-series arising from two semi-convergent factor-series of this kind will have, at regular finite intervals, terms $\sum a_{n-x} b_x$, composed of constituents $a_{n-x} b_x$ having all (or all excepting a finite number of them) the same sign, and the terms will vanish or not vanish as n increases indefinitely, according as the condition $r + s > 1$ is satisfied or not satisfied. In the latter case the product-series must diverge; in the former case it can be shown to converge by reasoning almost identical to that in the next paragraph.

§9. The condition $r + s > 1$ is *sufficient* to establish the convergence of the product of any two semi-convergent series, one of which becomes absolutely convergent on associating its terms into groups containing each a finite number of terms. In applying the criterion, choose r in II as large as possible, yet so that every term after a finite number of terms from the origin of one of the two given series is numerically equal to or less than the corresponding term in II. Choose s in $\sum \frac{(-1)^{n+1}}{n^s}$ in the same way with respect to the other given series. Then if $r + s > 1$, the product of the two given series is convergent.

In proving this statement, notice in the first place that the condition $r+s>1$ assures the vanishing of the n^{th} term in the product of $\sum \frac{(-1)^{n+1}}{n^r}$ and $\sum \frac{(-1)^{n+1}}{n^s}$, and since this n^{th} term is composed of constituents having all like signs, it follows that the n^{th} term of the product of the two given series will vanish also, since its constituents may have unlike signs and are never larger in absolute value (except perhaps in a finite number of cases) than the corresponding constituents in the n^{th} term of the product of $\sum \frac{(-1)^{n+1}}{n^r}$ and $\sum \frac{(-1)^{n+1}}{n^s}$. Moreover, the necessary and sufficient conditions I, for the case that the groups in one of the given semi-convergent series contain the same number p of terms, are also satisfied, for, besides the vanishing of the n^{th} and of all succeeding terms, they demand simply that the sum of certain constituents selected from a finite number $p - 1$ of successive terms after the $(n + 1)^{\text{th}}$ term in the product-series, shall vanish as n increases indefinitely.

* See E. Cesaro in Fortschritte der Mathematik, 1888, p. 243.

Nor are we, in the present instance, restricted to the case where the finite number of terms is *the same* in the groups in question: an examination of the mode of deriving the necessary and sufficient conditions* makes it plain that similar conditions hold true for the more general case.† These conditions are satisfied in the present instance.

§10. By reasoning like the above we can show that the condition which was obtained in §6 for the involution of the series II is a *sufficient* condition for the involution of any semi-convergent series which becomes absolutely convergent by associating its terms into groups containing each a finite number of terms. Choose r as large as possible, yet so that all the terms of the series to be tested are not greater in absolute value than the corresponding terms of II, then if $\frac{q-1}{q} < r$, the series can be raised to the positive integral power q by Cauchy's multiplication rule.

Example.—To what positive integral power can the semi-convergent series

$$\frac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \ldots .$$

be raised by Cauchy's multiplication rule? All the terms of the proposed series after the first term are numerically smaller than the corresponding terms in $\sum \frac{(-1)^{n+1}}{n}$. The latter series is obtained from II by putting $r = 1$, and can (§6) be raised to any finite positive integral power, since $r = 1 > \frac{q-1}{q}$ holds true for any value of q. Hence the proposed series can likewise be raised to any positive integral power.

§11. We proceed to extend the results of §9 to the multiplication of the two trigonometric series with real terms,

$$\sin\theta + \frac{1}{2^r}\sin 2\theta + \frac{1}{3^r}\sin 3\theta + \ldots + \frac{1}{n^r}\sin n\theta, \qquad \text{XIX}$$

$$\sin\phi + \frac{1}{2^s}\sin 2\phi + \frac{1}{3^s}\sin 3\phi + \ldots + \frac{1}{n^s}\sin n\phi, \qquad \text{XX}$$

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* Am. Jour. of Math., vol. XV, pp. 339–341. On p. 340, after the second expression for "E," insert the following: "*Similar expressions are gotten for the general case* $pm = 2ps + t$, *where* t *may have any value* 0, 1, 2, ... , $(2p - 1)$. *In any case a quantity* β," etc.

† These conditions are deduced in Bull. Am. Math. Soc., 2d series, vol. I, pp. 180–183.

If the angle of one of the series, say the angle θ, is commensurable with π so that $\theta = \frac{a\pi}{b}$, a and b being positive integers, then the series XIX becomes absolutely convergent on having its terms associated into groups containing $2b$ terms each. For observe that the b^{th} and the $2b^{\text{th}}$ term in each group vanish, while the other terms can be arranged in pairs so that in each pair the values of the sines are numerically the same but opposite in sign, while the difference, $\frac{1}{n^r} - \frac{1}{(n+t)^r}$, of the coefficients of the sines in each pair, where $t < 2b$, approaches, as n increases indefinitely, a limit not greater than $\frac{t}{n^{1+r}}$. Hence the sum of all the terms in one group approaches a limit not greater than $\frac{2b^2}{n^{1+r}}$. If $\therefore$ the series XIX, in which $\theta = \frac{a\pi}{b}$, be multiplied by series XX, then the product will be convergent whenever $r + s > 1$. This follows from the argument in §9.

That we may extend the above result to cases when θ is incommensurable with π, observe in the first place that the necessary and sufficient conditions for the product of two semi-convergent series, developed in the Am. Jour. of Math., vol. XV, p. 341, and restated in I, still hold true, if the number p of terms in each group is indefinitely great. Allowing $2b\,(=p)$ to increase indefinitely, but so that $2b < \log n$, the necessary and sufficient conditions for the convergence of the product-series are that each term after the $(n+1)^{\text{th}}$ term in that series approach the limit zero as n increases indefinitely, and that the sum of certain specified constituents selected from $2b - 1$ successive terms (the constituents in formula I) should also approach the limit zero. Now the $(2n - 1)^{\text{th}}$ term in the product of $\sum \frac{(-1)^{n+1}}{n^r}$ and $\sum \frac{(-1)^{n+1}}{n^s}$ is numerically less than XVIII. From the comparison of this expression with a similar one for the $(4n - 1)^{\text{th}}$ term, we conclude that the largest term between the $(2n - 1)^{\text{th}}$ and the $(4n - 1)^{\text{th}}$ terms is numerically less than

$$\frac{2^{(1-s)(m+2)}}{\left[2^{1-s} - 1\right] 2^{(m+1)r}} + \frac{2^{(1-r)(m+2)}}{\left[2^{1-r} - 1\right] 2^{(m+1)s}}.$$

If this value be multiplied by $2 \log n$, it is found that the product still approaches the limit zero as n increases indefinitely, as long as $r + s > 1$. Hence the sum

of the numerical values of $2(2b-1)$, $<2\log n$, successive terms after the $(2n-1)^{\text{th}}$ term in the product-series approaches the limit zero as n increases indefinitely. But all the constituents of which the terms in the product-series are composed have like signs in each term; hence zero is also the limit of the sum of any constituents selected from $2b-1$ successive terms after the $(2n-1)^{\text{th}}$ term in the product-series. This being the case, the product of XIX and XX converges even when $2b$ is indefinitely large, since XIX continues to be absolutely convergent when its terms are associated into groups of $2b$ terms in each group as long as $2b < \log n$, n increasing indefinitely, and since the terms in XIX and XX are numerically never greater than the corresponding terms in $\sum\frac{(-1)^{n+1}}{n^r}$ and $\sum\frac{(-1)^{n+1}}{n^s}$, thus assuring the vanishing, not only of the sum of $2(2b-1)$ terms after the $(2n-1)^{\text{th}}$ term in the product-series, but also the sum of any constituents selected from $2b-1$ of these terms.

The series XIX is *uniformly convergent* except in the neighborhood of $\theta = 0$ and $\theta = 2a\pi$. Similarly for XX and for those products of XIX and XX which we have proved to be convergent. Let β be a value of θ which is incommensurable with π. Choose a and b, so that $\beta > \frac{a\pi}{b}$ and $\beta < \frac{(a+1)\pi}{b}$. As b increases indefinitely, $\frac{a\pi}{b}$ approaches the limit β. Since the terms of the product of the series XIX and XX, when $\theta = \frac{a\pi}{b}$, are given for an indefinitely large number of values of θ within an interval $(\beta, \beta - \varepsilon)$, for which values β is the limit (said interval being taken as small as we please, though different from zero); since, moreover, the limiting values of those terms are definite and finite, and the product-series is uniformly convergent for the commensurable values of θ in that interval, it follows that the sum of the product-series has a definite finite limiting value for $\theta = \beta - 0$, and this limiting value is equal to the sum of the product-series when $\theta = \beta$. Hence the product-series is convergent when $\theta = \beta$,* and we have established that *the product of* XIX *and* XX *is convergent whenever* $r + s > 1$.

§12. The results of §11 may be generalized as follows: *If* $a_1, a_2, \ldots, a_n$ *is a series of positive terms such that both* $\sum(a_{2n} - a_{2n+1})$ and $\sum(a_{2n+1} - a_{2n+2})$

* Dini, Theorie der Functionen einer veränderlichen reellen Grösse, Leipzig, 1892, §94.

converge absolutely and at least as rapidly as does $\sum \frac{1}{n\lambda n\lambda^2 n \ldots . \lambda^{c-1}n\,(\lambda^c n)^{\kappa}}$, *where* $\kappa > 3$, *then the product of two convergent series*

$$a_1 \sin\theta + a_2 \sin 2\theta + \ldots . + a_n \sin n\theta, \qquad \text{XXI}$$

$$b_1 \sin\phi + b_2 \sin 2\phi + \ldots . + b_n \sin n\phi \qquad \text{XXII}$$

is convergent, provided that $r + s > 1$, *where* r *and* s *take the maximum values satisfying the relations* $a_n \overline{\overline{<}} \frac{1}{n^r}$ *and* $b_n \overline{\overline{<}} \frac{1}{n^s}$, *for all values of* n. By λ_n, $\lambda_n^2 \ldots .$ we here denote $\log n$, $\log\log n$, ... We first observe that, on the above assumptions, the series

$$(\pm a_1 \pm a_2 \pm \ldots . \pm a_{2b}) + (\pm a_{2b+1} \pm a_{2b+2} \pm \ldots . \pm a_{4b}) + \ldots .$$

is absolutely convergent whenever there are in each parenthesis as many positive terms as there are negative terms, and the order of the signs is the same in all parentheses. Suppose we desire to show that

$$\sum(a_{6n} + a_{6n+1} + a_{6n+2} - a_{6n+3} - a_{6n+4} - a_{6n+5})$$

is absolutely convergent. Add the terms in the corresponding parentheses of the following series which are easily seen to be absolutely convergent:

$$\sum(a_{6n} - a_{6n+1} + a_{6n+2} - a_{6n+3} + a_{6n+4} - a_{6n+5}),\ \sum(2a_{6n+3} - 2a_{6n+4}),$$
$$\sum(2a_{6n+2} - 2a_{6n+3}),\ \sum(2a_{6n+1} - 2a_{6n+2}),$$

and the sum, which is the required series, must be absolutely convergent. When the number $2b$ of terms in each parenthesis is indefinitely great, but less than $2\lambda^c n$, then the number of absolutely convergent series with binomial terms, which must be added in the above process, may be indefinitely great, but does not exceed $2\lambda^c n$. Since $\sum \frac{(2\lambda^c n)^2}{n\lambda n\lambda^2 n \ldots . \lambda^{c-1}n\,(\lambda^c n)^{\kappa}}$ is absolutely convergent, it follows that the sum is still an absolutely convergent series.

If, therefore, we take $\theta = \frac{a\pi}{b}$, we find that XXI becomes absolutely convergent if its terms are associated into groups of $2b$ terms in each. Hence, by

reasoning as in §11, we can establish the convergence of the product of XXI and XXII whenever $r+s>1$.

§13. A theorem corresponding to that in §10 holds for the series XIX and can be proved by the reasoning of §11, and observing that if the fraction in XIII(a) be multiplied by $2\log n$, it will still approach the limit zero, when $\frac{q-1}{q}<r$. Consequently, *the q^{th} power of series* XIX *is convergent whenever* $\frac{q-1}{q}<r$. In the same way we can show that *the q^{th} power of* XXI *is convergent whenever $\frac{q-1}{q}<r$, r being the maximum value satisfying the relation $a_n \lesseqgtr \frac{1}{n^r}$, for all values of n.*

§14. Proceeding to make a few extensions suggested by an article of A. Pringsheim,* we observe in the first place that, as $\sin n\theta = (-1)^n \sin n(\theta+\pi)$, it follows that our results on the multiplication of XIX and XX and of XXI and XXII, as well as our results on the involution of XIX and XXI, continue to hold true if the signs of the coefficients of the terms, instead of being all positive, are alternately plus and minus. In place of the points of non-uniform convergence $\theta=0$ and $\theta=2a\pi$, we have now the points $\theta=\pi$ and $\theta=(2a+1)\pi$.

§15. Our criterion for the applicability of Cauchy's multiplication rule, derived for XXI and XXII, applies also to two convergent series

$$a_1\cos\theta + a_2\cos 2\theta + \ldots + a_n\cos n\theta, \qquad \text{XXIII}$$
$$b_1\cos\phi + b_2\cos 2\phi + \ldots + b_n\cos n\phi. \qquad \text{XXIV}$$

If we make a few obvious alterations in the statements regarding places of non-uniform convergence and regarding values assumed by the trigonometric functions, then the reasoning of §§11, 12 will apply at once to XXIII and XXIV. In the same way the theorem on the involution of XXI applies to XXIII. Since $\cos n\theta = (-1)^n \cos n(\theta+\pi)$, the coefficients $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ may be alternately plus and minus instead of being all positive.

* Math. Annalen, XXVI, pp. 163–166.

§16. In the same way as was suggested in §15, we can show that the criterion for XXI and XXII applies to the product of two convergent series

$$a_1 \sin\theta + a_2 \sin 2\theta + \ldots . + a_n \sin n\theta,$$
$$b_1 \cos\phi + b_2 \cos 2\phi + \ldots . + b_n \cos n\phi,$$

no matter whether the coefficients in each series be all plus or whether they be alternately plus and minus.

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